MATH 245 S23, Exam 2 Solutions

1. Carefully define the following terms: Proof by Reindexed Induction, well-ordered by $<$.
To prove $\forall x \in \mathbb{N},\ P(x)$ by reindexed induction, we must (base case) prove $P(1)$ is true; and (inductive case) prove $\forall x \in \mathbb{N}$ (with $x \geq 2$), $P(x-1) \to P(x)$. Given a set S and an ordering $<$, we say that S is well-ordered by $<$ if every nonempty subset of S has a minimum element, in the $<$ ordering.

2. Carefully state the following: Proof by Cases Theorem, Proof by Contradiction Theorem
The Proof by Cases theorem states: For any propositions p, q, to prove $p \to q$, we can find propositions $c_1, \ldots c_k$ such that $c_1 \vee \cdots \vee c_k \equiv T$, and then prove $(p \wedge c_1) \to q, \ldots, (p \wedge c_k) \to q$. The Proof by Contradiction theorem states: For any propositions p, q, to prove $p \to q$, we can prove $p \wedge \neg q \equiv F$.

3. Let $a_n = 3n^2 + 7$. Prove that $a_n = \Theta(n^2)$.
This must be done in two parts, i.e. $a_n = O(n^2)$ and $a_n = \Omega(n^2)$.

   $a_n = \Omega(n^2)$: Choose $n_0 = 1$ and $M = 1$. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Now $M|a_n| = |3n^2 + 7| = 3n^2 + 7 \geq 3n^2 \geq n^2 = |n^2|$. Hence $M|a_n| \geq |n^2|$.

   NOTE: Some of you really wanted $n_0 = 1$ and $M = \frac{1}{10}$ for $a_n = \Omega(n^2)$. This doesn't work. For example, try $n = 2$: $a_n = 3 \cdot 2^2 + 7 = 19$, $n^2 = 2^2 = 4$, so $M|a_n| = \frac{19}{10} = 1.9 < 4 = |n^2|$.

   $a_n = O(n^2)$: Choose $n_0 = 3$ and $M = 4$. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Now $n^2 \geq 3^2 = 9 \geq 7$. Hence $3n^2 + 7 \leq 3n^2 + n^2$, so $|a_n| = |3n^2 + 7| = 3n^2 + 7 \leq 4n^2 = M|n^2|$. Hence $|a_n| \leq M|n^2|$.

   ALTERNATIVE $a_n = O(n^2)$: Choose $n_0 = 1$ and $M = 10$. Let $n \in \mathbb{N}$ with $n \geq n_0$ be arbitrary. Now $(M-3)n^2 = 7n^2 \geq 7$. Hence $3n^2 + 7 \leq Mn^2$, so $|a_n| = |3n^2 + 7| = 3n^2 + 7 \leq Mn^2 = M|n^2|$.

4. Prove the following: $\forall x \in \mathbb{R},\ 3x - 2|x+1| < x$.
Let $x \in \mathbb{R}$ be arbitrary. We have two cases, based on whether $x \geq -1$ or $x < -1$.
Case $x \geq -1$: Here $|x+1| = x+1$, so $3x - 2|x+1| = 3x - 2(x+1) = x - 2 < x$.
Case $x < -1$: Here $|x+1| = -(x+1)$, so $3x - 2|x+1| = 3x + 2(x+1) = 5x + 2$. Now, $x < -1$, so $4x < -4$, so $4x + 2 < -4 + 2 = -2 < 0$, so $x + 4x + 2 < x + 0$, so $5x + 2 < x$. Combining, we get $3x - 2|x+1| < x$, just as in the first case, as desired.

5. Prove the following: $\forall n \in \mathbb{N},\ \displaystyle\sum_{i=0}^{n}(2i-1) = n^2 - 1$.
Base case ($n = 1$): $\sum_{i=0}^{1}(2i-1) = -1 + 1 = 0$, which equals $1^2 - 1 = 0$.
Inductive case: Let $n \in \mathbb{N}$ and suppose that $\sum_{i=0}^{n}(2i-1) = n^2 - 1$. Add $2(n+1) - 1 = 2n+1$ to both sides, getting $\sum_{i=0}^{n+1}(2i-1) = n^2 - 1 + 2n + 1 = (n+1)^2 - 1$.

6. Recall the Fibonacci numbers F_n. Prove that $\forall n \in \mathbb{N}$ with $n \geq 3$, that $F_n > 1.1^n$.
Helpful facts: $1.1^2 = 1.21$, $1.1^3 = 1.331$, $1.1^4 = 1.4641$
We need two base cases: $F_3 = 2 > 1.331 = 1.1^3$, and $F_4 = 3 > 1.4641 = 1.1^4$.
Inductive case: Let $n \in \mathbb{N}$ with $n \geq 3$ be arbitrary. Assume that $F_n > 1.1^n$ and $F_{n+1} > 1.1^{n+1}$. Adding the inequalities, we get $F_{n+2} = F_n + F_{n+1} > 1.1^n + 1.1^{n+1} = 1.1^n(1+1.1) = 1.1^n(2.1) > 1.1^n(1.21) = 1.1^n 1.1^2 = 1.1^{n+2}$. Hence $F_{n+2} > 1.1^{n+2}$.

7. Solve the recurrence with initial conditions $a_0 = 2, a_1 = 3$, and recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$.

The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2$. Hence we have a double root $r = 1$, and the general solution is $a_n = A1^n + Bn1^n = A + Bn$. We now apply the initial conditions $2 = a_0 = A + B \cdot 0 = A$ and $3 = a_1 = A + B \cdot 1 = A + B$, to get $A = 2, B = 1$. Hence the specific solution is $a_n = 2 + n$.

For the remaining problems 8-10, we consider a new rounding function, "thround". For $x \in \mathbb{R}$, we define "the thround of x", writing $[x]$, as an integer satisfying $[x] - \frac{1}{3} \leq x < [x] + \frac{2}{3}$.

8. Prove uniqueness, i.e. $\forall x \in \mathbb{R}\ ![x] \in \mathbb{Z},\ [x] - \frac{1}{3} \leq x < [x] + \frac{2}{3}$.

Let $x \in \mathbb{R}$ be arbitrary. Suppose we had integers $[x]_1$ and $[x]_2$ satisfying $[x]_1 - \frac{1}{3} \leq x < [x]_1 + \frac{2}{3}$ and $[x]_2 - \frac{1}{3} \leq x < [x]_2 + \frac{2}{3}$. Combining the two orange inequalities gives $[x]_1 - \frac{1}{3} < [x]_2 + \frac{2}{3}$, i.e. $[x]_1 < [x]_2 + 1$ (after adding $\frac{1}{3}$ to both sides). Combining the other two inequalities gives $[x]_2 - \frac{1}{3} < [x]_1 + \frac{2}{3}$, i.e. $[x]_2 - 1 < [x]_1$ (after subtracting $\frac{2}{3}$ from both sides). Hence $[x]_2 - 1 < [x]_1 < x_2 + 1$, and by a theorem from the book (1.12d), we have $[x]_2 = [x]_1$.

9. Prove existence, i.e. $\forall x \in \mathbb{R}\ \exists [x] \in \mathbb{Z},\ [x] - \frac{1}{3} \leq x < [x] + \frac{2}{3}$.

We must begin by letting $x \in \mathbb{R}$ be arbitrary.

PROOF 1: Define $S = \{n \in \mathbb{Z} : n \leq x + \frac{1}{3}\}$, which has upper bound $x + \frac{1}{3}$. This is a half-line, so is a nonempty set of integers. By maximal element induction, there must be some maximum element $[x] \in S$ (in particular, $[x]$ is an integer). Hence, $[x] \leq x + \frac{1}{3}$ and $[x] + 1 > x + \frac{1}{3}$. Subtracting $\frac{1}{3}$ throughout and recombining, we get $[x] - \frac{1}{3} \leq x < [x] + \frac{2}{3}$.

PROOF 2: We apply the floor function to $x + \frac{1}{3}$, getting an integer $m = \lfloor x + \frac{1}{3} \rfloor$ which satisfies $m \leq x + \frac{1}{3} < m + 1$. Now, we subtract $\frac{1}{3}$ throughout, getting $m - \frac{1}{3} \leq x + \frac{1}{3} - \frac{1}{3} < m + 1 - \frac{1}{3}$, i.e. $m - \frac{1}{3} \leq x < m + \frac{2}{3}$. Hence we have found an integer, namely m, that satisfies the desired thround double inequality.

10. Prove or disprove: $\forall x \in \mathbb{R}\ \forall k \in \mathbb{Z},\ [x + k] = [x] + k$.

The statement is true. Let $x \in \mathbb{R}$ and $k \in \mathbb{Z}$ be arbitrary.

PROOF 1: Apply problem 9 to x to get $[x] - \frac{1}{3} \leq x < [x] + \frac{2}{3}$. Add k throughout to get $[x] + k - \frac{1}{3} \leq x + k < [x] + k + \frac{2}{3}$. Now, apply problem 8 to $x + k$. There is at most one integer n satisfying $n - \frac{1}{3} \leq x + k < n + \frac{2}{3}$. However, we have $n = [x] + k$ (from the preceding calculation) and $n = [x + k]$ (from problem 9 applied to $x + k$) satisfying both inequalities. Hence, $[x] + k = [x + k]$.

PROOF 2: Apply problem 9 to x to get $[x] - \frac{1}{3} \leq x < [x] + \frac{2}{3}$. Add k throughout to get $[x] + k - \frac{1}{3} \leq x + k < [x] + k + \frac{2}{3}$. Apply problem 9 to $x + k$ to get $[x + k] - \frac{1}{3} \leq x + k < [x + k] + \frac{2}{3}$. Combine the orange inequalities to get $[x] + k - \frac{1}{3} < [x + k] + \frac{2}{3}$, i.e. $[x] + k < [x + k] + 1$. Combine the two other inequalities to get $[x + k] - \frac{1}{3} < [x] + k + \frac{2}{3}$, i.e. $[x + k] - 1 < [x] + k$. Hence $[x + k] - 1 < [x] + k < [x + k] + 1$, so by a theorem from the book (1.12d), we have $[x + k] = [x] + k$.